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On the Kernel of Intersecting Families*

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Abstract. Let $\mathscr{F}$ be a t-wise s-intersecting family, i.e., $|F_1 \cap \cdots \cap F_t| \geq s$ holds for every t members of $\mathscr{F}$. Then there exists a set Y such that $|F_1 \cap \cdots \cap F_t \cap Y| \geq s$ still holds for every $F_1, \ldots, F_t \in \mathscr{F}$. Here exponential lower and upper bounds are proven for the possible sizes of Y.

1. Intersecting Families

A family of sets $\mathscr{F}$ (or a hypergraph) is called *intersecting* if $F \cap F' \neq \phi$ holds for all $F, F' \in \mathscr{F}$. The rank $r(\mathscr{F})$ is defined by

$$r(\mathscr{F}) =: \max\{|F| : F \in \mathscr{F}\}.$$

For a set Y define the restriction $\mathscr{F}|Y$ of $\mathscr{F}$ to Y by

$$\mathscr{F}|Y =: \{F \cap Y : F \in \mathscr{F}\}.$$

In 1964 Calczynska-Karlowicz [3] proved that for every k there exists an $n(k)$ such that for every intersecting hypergraph $\mathscr{F}$ of rank at most k there is a set Y of cardinality $n(k)$ such that $\mathscr{F}|Y$ is also intersecting. The first explicit upper bound for $n(k)$ was given by Ehrenfeucht and Mycielski [4]. Erdös and Lovász [5] proved $(2k-2) + \frac{1}{2}\binom{2k-2}{k-1} \leq n(k) \leq \frac{k}{2}\binom{2k-1}{k}$. The current best bounds are due to Tuza [15]:

$$(2k-4) + 2\binom{2k-4}{k-2} \leq n(k) \leq \binom{2k-1}{k-1} + \binom{2k-4}{k-2}. \tag{1}$$

Tuza's example giving the lower bound in (1) is the following: Let X be a $(2k-4)$-

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element set and for each partition $\{E, E'\}$ of X with $|E| = |E'| = k-2$, $E \cup E' = X$ take four new elements x, x', y, y' and let $E \cup \{x, y\}$, $E \cup \{x', y'\}$, $E' \cup \{x, x'\}$ and $E' \cup \{y, y'\}$ be members of $\mathscr{F}$. The obtained family $\mathscr{F}$ has $2\binom{2k-4}{k-2}$ members and if $\mathscr{F} | Y$ is intersecting then $Y \supseteq (\bigcup \mathscr{F})$ holds.

Conjecture 1.1 (Tuza [15]). For $k \geq 4$, $n(k) = (2k-4) + 2\binom{2k-4}{k-2}$.

Hansen and Toft [11] proved that $n(2) = 3$, $n(3) = 7$, $n(4) = 16$.

2. *t*-Wise *s*-Intersecting Families

A family $\mathscr{F}$ is *t-wise s-intersecting* if $|F_1 \cap F_2 \cap \cdots \cap F_t| \geq s$ holds whenever $F_1, \ldots, F_t \in \mathscr{F}$ $(t \geq 2, s \geq 1)$. In [8] the following generalization of Calczynska-Karlowicz's theorem was proved:

(2) If $\mathscr{F}$ is a t-wise s-intersecting family of k-sets then there exists a set Y, $|Y| \leq k^{2k}$, such that $|F_1 \cap \cdots \cap F_t \cap Y| \geq s$ still holds for every $F_1, \ldots, F_t \in \mathscr{F}$.

Denote by $n(k, t, s)$ the smallest integer n such that one can always find a Y, with $|Y| \leq n$, such that (2) remains true. With this notation the previous $n(k)$ is just $n(k, 2, 1)$. The existence of $n(k, t, s)$ was also proved by Frankl [6] (in an implicit form) and by Kahn and Seymour [12].

Define the *vertex-deleting* as the following operation on a family $\mathscr{F}$: suppose that x is a vertex covered by some members of $\mathscr{F}$, $\{x\} \notin \mathscr{F}$ and substitute all the edges $E \in \mathscr{F}$, $x \in E$ by $E - \{x\}$. A t-wise s-intersecting family is (t, s)-critical if it has no multiple edges and the hypergraph obtained by deleting any of its vertices is not t-wise s-intersecting. We can get a (t, s)-critical family from any t-wise s-intersecting $\mathscr{F}$ by deleting vertices as far as possible and deleting all but one copy of the appearing multiple edges. The obtained (smaller) family $\mathscr{K}$, is called the (t, s)-*kernel* of $\mathscr{F}$. (Of course, this $\mathscr{K}$ is not necessarily unique.) The following reformulation is obvious:

(3) $n(k, t, s) = \max\{|\bigcup \mathscr{K}| : \mathscr{K}$ is (t, s)-critical of rank at most $k\}$.

For $t \geq 3$ only a few (t, s)-critical hypergraphs are known, and each of them has less than k^2 vertices. The following Example shows that there are exponentially large (t, s)-critical hypergraphs.

Example 2.1. Define $\ell = [(k-s)/3(t-1)]$ and suppose that $\ell \geq 1$. We are going to construct a (t, s)-critical hypergraph $\mathscr{K}$ of rank $s + 3(t-1)\ell$ on $3^\ell + 3\ell t + s - 1$ vertices and with $(3t)^\ell$ edges. Let the vertex set consist of all the 3^ℓ sequences $\boldsymbol{a} = (a_1, \ldots, a_\ell)$, $a_i \in \{0, 1, 2\}$ together with an $(s-1)$-element set S and the disjoint union of ℓ $3t$-sets $C^1, \ldots, C^\ell$, where $C^i = \{x_0^i, x_1^i, \ldots, x_{3t-1}^i\}$. For each sequence $\mathbf{j} = (j_1, \ldots, j_\ell)$ with $0 \leq j_i < 3t$ we define an edge $E(\mathbf{j})$ of $\mathscr{K}$ by setting

$$E(\mathbf{j}) = S \cup \left(\bigcup_{1 \leq i \leq \ell} (C^i - \{x_{j_i}, x_{j_i+1}, x_{j_i+2}\}) \right) \cup \{\mathbf{r}\}$$

where the indices are reduced modulo $3t$ and where $\mathbf{r} = (r_1, \ldots, r_\ell)$ is defined by $r_i \equiv j_i \pmod 3$.

It is left to the reader to check that $\mathscr{K}$ is (t, s)-critical.

Now the following theorem implies that

$$\frac{1}{3}(3^{1/3})^{(k-s)/(t-1)} \le n(k,t,s) < k\left[\frac{t^t}{(t-1)^{t-1}}\right]^{[(k-s)/(t-1)]} < k(et)^{(k-s)/(t-1)}. \tag{4}$$

Theorem 2.2. *Let $\mathscr{K}$ be a (t, s)-critical family of sets having size at most k. Then*

$$\left|\bigcup \mathscr{K}\right| \le k \binom{k - s + 1 + \left\lceil \frac{k-s}{t-1} \right\rceil}{\left\lceil \frac{k-s}{t-1} \right\rceil}.$$

Proof. We are going to use the following result:

(5) Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be finite sets with the following properties: $|A_i| \le a$, $|B_i| \le b$, $|A_i \cap B_i| \le c$ and $|A_i \cap B_j| > c$ for all $1 \le i < j \le m$. Then $m \le \binom{a+b-2c}{a-c}$.

This theorem was conjectured by Frankl and Stečkin [9] and proved in [10]. The case $c = 0$ was proved by Frankl [7] and Kalai [13]. See also [1], [2], and [14].

Let x be a vertex of $\mathscr{K}$. Then $\mathscr{K} - \{x\}$ is not t-wise s-interesting, so there exist $H_1, \ldots, H_t \in \mathscr{K}$ such that $x \in \bigcap H_i$, $|\bigcap H_i| = s$ $(1 \le i \le t)$. The sets $(\bigcap_{i \ne j} H_i) - (\bigcap H_i)$ are pairwise disjoint and for $1 \le j \le t-1$ they are included in $H_t - (\bigcap H_i)$. Hence there exists a j $(1 \le j \le t-1)$ such that $|\bigcap_{i \ne j} H_i - \bigcap H_i| \le [(k-s)/(t-1)]$. Define

$$H(x) =: H_j \quad \text{and}$$
$$S(x) =: \bigcap \{H_i : i \ne j, 1 \le i \le t\} - \{x\}.$$

Define a sequence $x_1, \ldots, x_m$ as follows. Choose x_1 arbitrarily from $\bigcup \mathscr{K}$ and if $x_1, \ldots, x_{i-1}$ are chosen then let $x_i \in (\bigcup \mathscr{K}) - (\bigcup \{H(x_j) : j < i\})$. Stop if $\bigcup\{H(x_j) : j \le i\} = \bigcup \mathscr{K}$. Then we can use (5) with $A_i = H(x_i)$, $B_i = S(x_i)$, $a = k$, $b = s - 1 + [(k-s)/(t-1)]$, $c = s - 1$. We obtain:

$$\left|\bigcup \mathscr{K}\right| = \left|\bigcup_i H(x_i)\right| \le \sum_i |H(x_i)| \le k\binom{a+b-2c}{a-c},$$

implying Theorem 2.2. Finally, standard computation gives (4). (E.g., one can use that for all $a, b \ge 1$ we have

$$\binom{a+b}{a} \le \frac{(a+b)^{a+b}}{a^a b^b}\sqrt{\frac{a+b}{2\pi ab}}.)$$

3. Problems

In many extremal problems using the kernel of an intersecting family is very fruitful, so it would be interesting to narrow the gap between the lower and upper bounds of $n(k, t, s)$. The most important case is when $s = 1$. The authors strongly believe that the following is true.

Conjecture 3.1. If t is fixed, and k tends to infinity then $\lim_{k\to\infty} \sqrt[k]{n(k, t, 1)}$ exists.

(By (4) the value of this limit is between $3^{1/3(t-1)}$ and $(t/(t-1))t^{1/(t-1)}$.) Lovász [5], Tuza [15] and Hanson and Toft [11] introduced other versions of the kernel (not only for intersecting families) which also deserve further investigations.

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